

A Note on the Construction of Complex and Quaternionic Vector Fields on Spheres *

Mohammad Obiedat

Department of Science, Technology, and Mathematics,
Gallaudet University, 800 Florida Avenue NE, Washington,
DC 20002-3695, USA[†]

May 31, 2016

Abstract

A relationship between real, complex, and quaternionic vector fields on spheres is given by using a relationship between the corresponding standard inner products. The number of linearly independent complex vector fields on the standard $(4n - 1)$ -sphere is shown to be twice the number of linearly independent quaternionic vector fields plus d , where $d = 1$ or 3 .

2010 MSC: Primary 57R25; Secondary 19L20, 55Q50.

Keywords. Complex vector fields, Quaternionic vector fields, Realification function, Complexification function, James numbers.

1 Introduction

Let \mathbb{F} be one of the associative division algebras over \mathbb{R} , namely, the real numbers \mathbb{R} , the complex numbers $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$, or the quaternionic

*Published in the Journal of Mathematical Notes, 93(1) (2013), 104-110

[†]Email address: mohammad.obiedat@gallaudet.edu

numbers $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$, with the usual norm and conjugation operations. For each $n \in \mathbb{N}$, we consider \mathbb{F}^n as a right inner product space over \mathbb{F} with the usual multiplication and the standard \mathbb{F} -inner product $\langle \cdot | \cdot \rangle_{\mathbb{F}} : \mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}$ given by $\langle (x_1, \dots, x_n) | (y_1, \dots, y_n) \rangle_{\mathbb{F}} = \sum_{m=1}^n \bar{x}_m y_m$. Let $S(\mathbb{F}^n) = \{x \in \mathbb{F}^n : \langle x | x \rangle_{\mathbb{F}} = 1\}$ be the standard unit sphere in \mathbb{F}^n . An \mathbb{F} -vector field on $S(\mathbb{F}^n)$ is a continuous function $v : S(\mathbb{F}^n) \rightarrow \mathbb{F}^n - \{0\}$ such that $\langle x | v(x) \rangle_{\mathbb{F}} = 0$ for each $x \in S(\mathbb{F}^n)$. Given m such \mathbb{F} -vector fields v_1, \dots, v_m on $S(\mathbb{F}^n)$, we say that they are linearly independent if the vectors $v_1(x), \dots, v_m(x)$ are linearly independent over \mathbb{F} for all $x \in S(\mathbb{F}^n)$. Let $\rho^{\mathbb{F}}(\mathbb{F}^n)$ denote the maximal number of linearly independent \mathbb{F} -vector fields on $S(\mathbb{F}^n)$. The vector fields on spheres problem has two sides: the first side is what we call the maximal number problem, namely, the computation of $\rho^{\mathbb{F}}(\mathbb{F}^n)$, the second side is what we call the construction problem, namely, the actual construction of $\rho^{\mathbb{F}}(\mathbb{F}^n)$ linearly independent \mathbb{F} -vector fields on $S(\mathbb{F}^n)$. The roots of this classical problem goes back to the hairy ball theorem, the parallelizability of spheres, and the classification of division algebras over \mathbb{R} . For background materials on this problem, we refer the reader to [5, 10].

If n is odd, then one can easily show that $\rho^{\mathbb{F}}(\mathbb{F}^n) = 0$. Therefore, unless otherwise indicated, we will assume that n is an even positive integer. An explicit formula for computing $\rho^{\mathbb{R}}(\mathbb{R}^n)$ was given by Adams [1] in 1962, when he showed that $\rho^{\mathbb{R}}(\mathbb{R}^n) = \rho(n) - 1$, where $\rho(n)$ be the Radon-Hurwitz function. While there are no explicit formulas for computing $\rho^{\mathbb{C}}(\mathbb{C}^n)$ and $\rho^{\mathbb{H}}(\mathbb{H}^n)$, they still can be obtained by using the work of Adams and Walker [2] in 1965 for the complex case, and the work of Sigrist and Suter [9] in 1973 for the quaternionic case. More direct formulas for computing $\rho^{\mathbb{C}}(\mathbb{C}^n)$ and $\rho^{\mathbb{H}}(\mathbb{H}^n)$ are given in Section 3 of this paper.

Our main goal in this paper is to draw more attention to the importance of the construction problem of vector fields on spheres. There are several known methods that explicitly give $\rho^{\mathbb{R}}(\mathbb{R}^n)$ linearly independent real vector fields on $S(\mathbb{R}^n)$ (e.g., see [7, 11]). The situation is completely different with the construction of complex and quaternionic vector fields; there is no explicit construction that gives two or more linearly independent complex vector fields on $S(\mathbb{C}^n)$, and there is no known construction that gives even a single quaternionic vector field on $S(\mathbb{H}^n)$. In addition to their self importance, the actual construction of complex and quaternionic vector fields on spheres might lead to the solution of several open problems in the equivariant complex and quaternionic vector fields on spheres (see [6, 8]).

The layout of this paper is as follows. In Section 2, we present a relation-

ship between real, complex, and quaternionic standard inner products. We show how such a relationship leads to a relationship between corresponding real, complex, and quaternionic vector fields on spheres. More specifically, we show how m linearly independent quaternionic vector fields can be used to obtain $2m$ linearly independent complex vector fields, and how m linearly independent complex vector fields can be used to obtain $2m$ linearly independent real vector fields. In Section 3, we provide direct formulas for computing $\rho^{\mathbb{C}}(\mathbb{C}^n)$ and $\rho^{\mathbb{H}}(\mathbb{H}^n)$, and show that $\rho^{\mathbb{C}}(\mathbb{C}^{2n}) = 2\rho^{\mathbb{H}}(\mathbb{H}^n) + d$ where $d = 1$ or 3 . In Section 4, we give necessary and sufficient conditions on linearly independent real (respectively, complex) vector fields to be linearly independent complex or quaternionic (respectively, quaternionic) vector fields.

2 A relationship between real, complex, and quaternionic vector fields on spheres

For any positive integer n , let $r_{\mathbb{C}} : \mathbb{C}^n \rightarrow \mathbb{R}^{2n}$ be the realification function from \mathbb{C}^n to \mathbb{R}^{2n} defined by $r_{\mathbb{C}}(a_1 + b_1i, \dots, a_n + b_ni) = (a_1, b_1, \dots, a_n, b_n)$, $c_{\mathbb{H}} : \mathbb{H}^n \rightarrow \mathbb{C}^{2n}$ be the complexification function from \mathbb{H}^n to \mathbb{C}^{2n} defined by $c_{\mathbb{H}}(a_1 + b_1i + c_1j + d_1k, \dots, a_n + b_ni + c_nj + d_nk) = (a_1 + b_1i, d_1 + c_1i, \dots, a_n + b_ni, d_n + c_ni)$, and $r_{\mathbb{H}} : \mathbb{H}^n \rightarrow \mathbb{R}^{4n}$ be the realification function from \mathbb{H}^n to \mathbb{R}^{4n} defined by $r_{\mathbb{H}} = r_{\mathbb{C}} \circ c_{\mathbb{H}}$. For $t \in \mathbb{F}$, let $\alpha_t : \mathbb{F}^n \rightarrow \mathbb{F}^n$ be the right multiplication by t , i.e., $\alpha_t(x) = x \cdot t$ for each $x \in \mathbb{F}^n$. The proof of the following lemma is straightforward.

Lemma 2.1 *Let $s \in \mathbb{C}$ and $t \in \mathbb{H}$. Then*

- (i) $c_{\mathbb{H}} \circ \alpha_s = \alpha_s \circ c_{\mathbb{H}}$.
- (ii) $r_{\mathbb{C}} \circ \alpha_s \circ c_{\mathbb{H}} = r_{\mathbb{H}} \circ \alpha_s$.
- (iii) $r_{\mathbb{C}} \circ \alpha_s \circ c_{\mathbb{H}} \circ \alpha_t = r_{\mathbb{H}} \circ \alpha_{ts}$.

In the following theorem, we give a relationship between real, complex, and quaternionic standard inner products on \mathbb{F}^n .

Theorem 2.2 *Let $x, y \in \mathbb{C}^n$ and $v, w \in \mathbb{H}^n$. Then*

- (i) $\langle x|y \rangle_{\mathbb{C}} = \langle r_{\mathbb{C}}(x)|r_{\mathbb{C}}(y) \rangle_{\mathbb{R}} - \langle r_{\mathbb{C}}(x)|r_{\mathbb{C}} \circ \alpha_i(y) \rangle_{\mathbb{R}}i$.

- (ii) $\langle v|w \rangle_{\mathbb{H}} = \langle c_{\mathbb{H}}(v)|c_{\mathbb{H}}(w) \rangle_{\mathbb{C}} - \langle c_{\mathbb{H}}(v)|c_{\mathbb{H}} \circ \alpha_j(w) \rangle_{\mathbb{C}} j.$
- (iii) $\langle v|w \rangle_{\mathbb{H}} = \langle r_{\mathbb{H}}(v)|r_{\mathbb{H}}(w) \rangle_{\mathbb{R}} - \langle r_{\mathbb{H}}(v)|r_{\mathbb{H}} \circ \alpha_i(w) \rangle_{\mathbb{R}} i$
 $- \langle r_{\mathbb{H}}(v)|r_{\mathbb{H}} \circ \alpha_j(w) \rangle_{\mathbb{R}} j - \langle r_{\mathbb{H}}(v)|r_{\mathbb{H}} \circ \alpha_k(w) \rangle_{\mathbb{R}} k.$

Proof. Without loss of generality, we can assume that $n = 1$. Then the result follows by using Lemma 2.1 and the definition of the standard inner products.

In the following theorem, we give a relationship between real, complex, and quaternionic vector fields on spheres.

Theorem 2.3 *Let $v : S(\mathbb{C}^n) \rightarrow \mathbb{C}^n - \{0\}$ and $w : S(\mathbb{H}^n) \rightarrow \mathbb{H}^n - \{0\}$ be two continuous functions. Then*

- (i) *v is a complex vector field on $S(\mathbb{C}^n)$ if and only if $r_{\mathbb{C}} \circ v \circ r_{\mathbb{C}}^{-1}$ and $r_{\mathbb{C}} \circ \alpha_i \circ v \circ r_{\mathbb{C}}^{-1}$ are real vector fields on $S(\mathbb{R}^{2n})$.*
- (ii) *w is a quaternionic vector field on $S(\mathbb{H}^n)$ if and only if $c_{\mathbb{H}} \circ w \circ c_{\mathbb{H}}^{-1}$ and $c_{\mathbb{H}} \circ \alpha_j \circ w \circ c_{\mathbb{H}}^{-1}$ are complex vector fields on $S(\mathbb{C}^{2n})$.*
- (iii) *w is a quaternionic vector field on $S(\mathbb{H}^n)$ if and only if $r_{\mathbb{H}} \circ w \circ r_{\mathbb{H}}^{-1}$ and $r_{\mathbb{H}} \circ \alpha_t \circ w \circ r_{\mathbb{H}}^{-1}$, where $t \in \{i, j, k\}$, are real vector fields on $S(\mathbb{R}^{4n})$.*

Proof. (i) Let $x \in S(\mathbb{R}^{2n})$. By Theorem 2.2 (i), $\langle r_{\mathbb{C}}^{-1}(x)|v(r_{\mathbb{C}}^{-1}(x)) \rangle_{\mathbb{C}} = 0$ if and only if $\langle x|r_{\mathbb{C}} \circ v \circ r_{\mathbb{C}}^{-1}(x) \rangle_{\mathbb{R}} = 0$ and $\langle x|r_{\mathbb{C}} \circ \alpha_i \circ v \circ r_{\mathbb{C}}^{-1}(x) \rangle_{\mathbb{R}} = 0$. The result follows. Similarly, one can prove (ii) and (iii) by using Theorem 2.2 (ii) and (iii).

Example 2.4 For each $n \geq 2$, $v : S(\mathbb{C}^n) \rightarrow \mathbb{C}^n - \{0\}$ such that $v(x_1 + ix_2, \dots, x_{2n-1} + ix_{2n}) = (-x_3 + ix_4, x_1 - ix_2, \dots, -x_{2n-1} + ix_{2n}, x_{2n-3} - ix_{2n-2})$ is a complex vector field on $S(\mathbb{C}^n)$. By Theorem 2.3 (i), $r_{\mathbb{C}} \circ v \circ r_{\mathbb{C}}^{-1}$ and $r_{\mathbb{C}} \circ \alpha_i \circ v \circ r_{\mathbb{C}}^{-1}$ are real vector fields on $S(\mathbb{R}^{2n})$. $r_{\mathbb{C}} \circ v \circ r_{\mathbb{C}}^{-1}$ is given by $(x_1, \dots, x_{2n}) \mapsto (-x_3, x_4, x_1, -x_2, \dots, -x_{2n-1}, x_{2n}, x_{2n-3}, -x_{2n-2})$ and $r_{\mathbb{C}} \circ \alpha_i \circ v \circ r_{\mathbb{C}}^{-1}$ is given by $(x_1, \dots, x_{2n}) \mapsto (-x_4, -x_3, x_2, x_1, \dots, -x_{2n}, -x_{2n-1}, x_{2n-2}, x_{2n-3})$.

Theorem 2.5 *Let $v_1, \dots, v_m : S(\mathbb{C}^n) \rightarrow \mathbb{C}^n - \{0\}$ and $w_1, \dots, w_m : S(\mathbb{H}^n) \rightarrow \mathbb{H}^n - \{0\}$ be continuous functions. Then*

- (i) v_1, \dots, v_m are linearly independent complex vector field on $S(\mathbb{C}^n)$ if and only if $r_{\mathbb{C}} \circ v_1 \circ r_{\mathbb{C}}^{-1}, r_{\mathbb{C}} \circ \alpha_i \circ v_1 \circ r_{\mathbb{C}}^{-1}, \dots, r_{\mathbb{C}} \circ v_m \circ r_{\mathbb{C}}^{-1}, r_{\mathbb{C}} \circ \alpha_i \circ v_m \circ r_{\mathbb{C}}^{-1}$ are linearly independent real vector fields on $S(\mathbb{R}^{2n})$.
- (ii) w_1, \dots, w_m are linearly independent quaternionic vector field on $S(\mathbb{H}^n)$ if and only if $c_{\mathbb{H}} \circ w_1 \circ c_{\mathbb{H}}^{-1}, c_{\mathbb{H}} \circ \alpha_j \circ w_1 \circ c_{\mathbb{H}}^{-1}, \dots, c_{\mathbb{H}} \circ w_m \circ c_{\mathbb{H}}^{-1}, c_{\mathbb{H}} \circ \alpha_j \circ w_m \circ c_{\mathbb{H}}^{-1}$ are linearly independent complex vector fields on $S(\mathbb{C}^{2n})$.
- (iii) w_1, \dots, w_m are linearly independent quaternionic vector field on $S(\mathbb{H}^n)$ if and only if $r_{\mathbb{H}} \circ w_1 \circ r_{\mathbb{H}}^{-1}, r_{\mathbb{H}} \circ \alpha_t \circ w_1 \circ r_{\mathbb{H}}^{-1}, \dots, r_{\mathbb{H}} \circ w_m \circ r_{\mathbb{H}}^{-1}, r_{\mathbb{H}} \circ \alpha_t \circ w_m \circ r_{\mathbb{H}}^{-1}$, where $t \in \{i, j, k\}$, are linearly independent real vector fields on $S(\mathbb{R}^{4n})$.

Proof. (i) By Theorem 2.3 (i), v_1, \dots, v_m are complex vector fields on $S(\mathbb{C}^n)$ if and only if $r_{\mathbb{C}} \circ v_1 \circ r_{\mathbb{C}}^{-1}, r_{\mathbb{C}} \circ \alpha_i \circ v_1 \circ r_{\mathbb{C}}^{-1}, \dots, r_{\mathbb{C}} \circ v_m \circ r_{\mathbb{C}}^{-1}, r_{\mathbb{C}} \circ \alpha_i \circ v_m \circ r_{\mathbb{C}}^{-1}$ are real vector fields on $S(\mathbb{R}^{2n})$. Let $x \in S(\mathbb{R}^{2n}), y = r_{\mathbb{C}}^{-1}(x)$, and $a_1, b_1, \dots, a_m, b_m \in \mathbb{R}$. Then $(r_{\mathbb{C}} \circ v_1 \circ r_{\mathbb{C}}^{-1}(x))a_1 + (r_{\mathbb{C}} \circ \alpha_i \circ v_1 \circ r_{\mathbb{C}}^{-1}(x))b_1 + \dots + (r_{\mathbb{C}} \circ v_m \circ r_{\mathbb{C}}^{-1}(x))a_m + (r_{\mathbb{C}} \circ \alpha_i \circ v_m \circ r_{\mathbb{C}}^{-1}(x))b_m = 0$ if and only if $v_1(y)(a_1 + b_1 i) + \dots + v_m(y)(a_m + b_m i) = 0$. The result follows. (ii) and (iii) are similar to (i).

Corollary 2.6 $\rho^{\mathbb{R}}(\mathbb{R}^{2n}) \geq 2\rho^{\mathbb{C}}(\mathbb{C}^n)$ and $\rho^{\mathbb{C}}(\mathbb{C}^{2n}) \geq 2\rho^{\mathbb{H}}(\mathbb{H}^n)$.

Remark 2.7 A unit \mathbb{F} -vector field on $S(\mathbb{F}^n)$ is an \mathbb{F} -vector field on $S(\mathbb{F}^n)$ whose image is a subset of $S(\mathbb{F}^n)$. m such unit vector fields v_1, \dots, v_m are orthonormal if $\langle v_s(x) | v_t(x) \rangle_{\mathbb{F}} = \delta_{st}$ for each $s, t \in \{1, \dots, m\}$ and all $x \in S(\mathbb{F}^n)$. Observe that since the Gram-Schmidt orthonormalization process is continuous, then one can convert m linearly independent \mathbb{F} -vector fields on $S(\mathbb{F}^n)$ to m orthonormal \mathbb{F} -vector fields on $S(\mathbb{F}^n)$. Consequently, $\rho^{\mathbb{F}}(\mathbb{F}^n)$ is equal to the maximal number of orthonormal \mathbb{F} -vector fields on $S(\mathbb{F}^n)$. Also, one can use Theorem 2.2 and basic properties of standard inner products to prove that Theorem 2.3 remains valid if we replace “vector field” by “unit vector field”, and Theorem 2.5 remains valid if we replace “linearly independent” by “orthonormal”.

3 A relationship between $\rho^{\mathbb{H}}(\mathbb{H}^n)$ and $\rho^{\mathbb{C}}(\mathbb{C}^{2n})$

For a positive integer m , let $c_m^{\mathbb{F}}$ be the \mathbb{F} -James number mentioned in [4]. Then $\rho^{\mathbb{F}}(\mathbb{F}^n)$ is the largest m such that n is a multiple of $c_m^{\mathbb{F}}$. $c_m^{\mathbb{R}}$ is computed

by Adams in [1] and is given by $c_m^{\mathbb{R}} = 2^{f(m)}$ where $f(m)$ is the number of integers q such that $0 < q \leq m$ and $q \equiv 0, 1, 2$ or $4 \pmod{8}$. In [1], Adams also showed that $\rho^{\mathbb{R}}(\mathbb{R}^n) = 8d + 2^c - 1$ where $n = (2a + 1)2^{4d+c}$ for some non-negative integers a, d, c with $0 \leq c \leq 3$.

Now, we show how to compute $\rho^{\mathbb{C}}(\mathbb{C}^n), \rho^{\mathbb{H}}(\mathbb{H}^n)$. For a given prime p , let $\nu_p(m)$ be the exponent of p in the prime decomposition of m . $c_m^{\mathbb{C}}$ is computed by Adams-Walker [2] and is given by

$$\nu_p(c_m^{\mathbb{C}}) = \max\{s + \nu_p(s) : 0 \leq s \leq \lfloor \frac{m}{p-1} \rfloor\}.$$

$c_m^{\mathbb{H}}$ is computed by Sigrist-Suter [9] and is given by $\nu_2(c_m^{\mathbb{H}}) = \max\{2m + 1, 2s + \nu_2(s) : 0 \leq s \leq m\}$, and for an odd prime p ,

$$\nu_p(c_m^{\mathbb{H}}) = \max\{s + \nu_p(s) : 0 \leq s \leq \lfloor \frac{2m}{p-1} \rfloor\}.$$

For $p \leq m + 1$, let

$$S_{m,p} = \{s : \lfloor \frac{m}{p-1} \rfloor + \nu_p(\lfloor \frac{m}{p-1} \rfloor) - \lfloor \log_p(\frac{m}{p-1}) \rfloor \leq s \leq \lfloor \frac{m}{p-1} \rfloor\},$$

and for $p > m + 1$, let $S_{m,p} = \{0\}$. Notice that if $p \leq m + 1$, then $|S_{m,p}| = \lfloor \log_p(m/(p-1)) \rfloor - \nu_p(\lfloor m/(p-1) \rfloor) + 1$. In the following lemma, we give refined formulas for computing $\nu_p(c_m^{\mathbb{C}})$ and $\nu_p(c_m^{\mathbb{H}})$.

Lemma 3.1 *Let m be a positive integer and p be a prime number. Then*

- (i) $\nu_p(c_m^{\mathbb{C}}) = \max\{s + \nu_p(s) : s \in S_{m,p}\}$.
- (ii) $\nu_2(c_m^{\mathbb{H}}) = \max\{2m + 1, 2s + \nu_2(s) : s \in S_{m,2}\}$, and for an odd prime p ,
 $\nu_p(c_m^{\mathbb{H}}) = \max\{s + \nu_p(s) : s \in S_{2m,p}\}$.

Proof. (i) Let $\nu_p(c_m^{\mathbb{C}}) = u + \nu_p(u)$ for some $0 \leq u \leq \lfloor m/(p-1) \rfloor$. Then $\nu_p(u) \leq \lfloor \log_p u \rfloor \leq \lfloor \log_p(m/(p-1)) \rfloor$. So, $\lfloor m/(p-1) \rfloor + \nu_p(\lfloor m/(p-1) \rfloor) \leq u + \nu_p(u) \leq u + \lfloor \log_p(m/(p-1)) \rfloor$. Hence, $u \geq \lfloor m/(p-1) \rfloor + \nu_p(\lfloor m/(p-1) \rfloor) - \lfloor \log_p(m/(p-1)) \rfloor$. The result follows. (ii) is similar to (i).

Let p_i be the i th prime number, i.e., $p_1 = 2, p_2 = 3, p_3 = 5$, etc. Let $n = p_1^{t_1} \times \cdots \times p_r^{t_r} \times l$, where $t_i \geq 1$ for each $i = 1, \dots, r$ and $\nu_{p_{r+1}}(l) = 0$. For each $i = 1, \dots, r$, let $K_{n,i}^{\mathbb{C}}$ be the largest $m \in \{0, \dots, p_{r+1} - 2\}$ such that $t_i \geq \nu_{p_i}(c_m^{\mathbb{C}})$, and let $K_{n,i}^{\mathbb{H}}$ be the largest $m \in \{0, \dots, (p_{r+1} - 3)/2\}$ such that $t_i \geq \nu_{p_i}(c_m^{\mathbb{H}})$. In the following theorem, we give direct formulas for computing $\rho^{\mathbb{C}}(\mathbb{C}^n)$ and $\rho^{\mathbb{H}}(\mathbb{H}^n)$.

Theorem 3.2 Let $n = p_1^{t_1} \times \cdots \times p_r^{t_r} \times l$, where $t_i \geq 1$ for each $i = 1, \dots, r$ and $\nu_{p_{r+1}}(l) = 0$. Then

$$(i) \quad \rho^{\mathbb{C}}(\mathbb{C}^n) = \min\{K_{n,i}^{\mathbb{C}} : i = 1, \dots, r\}.$$

$$(ii) \quad \rho^{\mathbb{H}}(\mathbb{H}^n) = \min\{K_{n,i}^{\mathbb{H}} : i = 1, \dots, r\}.$$

Proof. (i) First, since $\nu_{p_{r+1}}(c_{p_{r+1}-1}^{\mathbb{C}}) = 1$ and $c_i^{\mathbb{C}} \mid c_{i+1}^{\mathbb{C}}$ for each $i \geq 1$ then $\rho^{\mathbb{C}}(\mathbb{C}^n) \leq K_{n,i}^{\mathbb{C}}$ for each $i = 1, \dots, r$. On the other hand, if $s = \min\{K_{n,i}^{\mathbb{C}} : i = 1, \dots, r\}$ then $t_i \geq \nu_{p_i}(c_s^{\mathbb{C}})$ for each $i = 1, \dots, r$, and hence $\rho^{\mathbb{C}}(\mathbb{C}^n) \geq s$. The result follows. (ii) is similar to (i).

From Corollary 2.6, we know that the number of linearly independent complex vector fields on $S(\mathbb{C}^n)$ is at most half the number of linearly independent real vector fields on $S(\mathbb{R}^{2n})$, and the number of linearly independent quaternionic vector fields on $S(\mathbb{H}^n)$ is at most half the number of linearly independent complex vector fields on $S(\mathbb{C}^{2n})$. The main result of this section is the following theorem, which gives an explicit relationship between the number of linearly independent complex vector fields on $S(\mathbb{C}^{2n})$ and the number of linearly independent quaternionic vector fields on $S(\mathbb{H}^n)$.

Theorem 3.3 $\rho^{\mathbb{C}}(\mathbb{C}^{2n}) = 2\rho^{\mathbb{H}}(\mathbb{H}^n) + d$ where $d = 1$ or 3 .

Proof. Let $\rho^{\mathbb{H}}(\mathbb{H}^n) = m$. Then m is the smallest integer such that n is not a multiple of $c_{m+1}^{\mathbb{H}}$. From [9], $c_{m+1}^{\mathbb{H}}$ is either equal to $c_{2m+3}^{\mathbb{C}}/2$ or to $c_{2m+3}^{\mathbb{C}}$. If $c_{m+1}^{\mathbb{H}} = c_{2m+3}^{\mathbb{C}}/2$, then $2n$ is not a multiple of $2c_{m+1}^{\mathbb{H}} = c_{2m+3}^{\mathbb{C}}$. On the other hand, if $c_{m+1}^{\mathbb{H}} = c_{2m+3}^{\mathbb{C}}$ then $\nu_2(c_{m+1}^{\mathbb{H}}) = \nu_2(c_{2m+3}^{\mathbb{C}}) = 2m + 3$. Hence, $2n$ is not a multiple of $c_{m+2}^{\mathbb{H}}$, because $c_{m+2}^{\mathbb{H}} \geq c_{m+1}^{\mathbb{H}}$, $\nu_p(2n) = \nu_p(n)$ for each $p \neq 2$, and $\nu_2(c_{m+2}^{\mathbb{H}}) \geq 2m + 5$, see Lemma 3.1. But, $c_{m+2}^{\mathbb{H}}$ is either equal to $c_{2m+5}^{\mathbb{C}}/2$ or to $c_{2m+5}^{\mathbb{C}}$. Thus, in all cases, $2n$ is not a multiple of $c_{2m+5}^{\mathbb{C}}$, and hence $\rho^{\mathbb{C}}(\mathbb{C}^{2n}) < 2m + 5$. Now, since $c_{2k+1}^{\mathbb{C}} = c_{2k}^{\mathbb{C}}$ for each $k \geq 1$, see [2], then $\rho^{\mathbb{C}}(\mathbb{C}^{2n})$ is odd. Consequently, by Corollary 2.6, $\rho^{\mathbb{C}}(\mathbb{C}^{2n})$ is either equal to $2m + 1$ or to $2m + 3$.

In the following table, we give the values of $\rho^{\mathbb{R}}(\mathbb{R}^{4n})$, $\rho^{\mathbb{C}}(\mathbb{C}^{2n})$, and $\rho^{\mathbb{H}}(\mathbb{H}^n)$ for some values of n .

4 Construction of vector fields on spheres

The construction of $\rho^{\mathbb{R}}(\mathbb{R}^n)$ linearly independent real vector fields on $S(\mathbb{R}^n)$ is well understood, for constructions using Clifford algebras see [11], and for

n	1	2	4	6	12	24	1440
$\rho^{\mathbb{R}}(\mathbb{R}^{4n})$	3	7	8	7	8	9	15
$\rho^{\mathbb{C}}(\mathbb{C}^{2n})$	1	1	1	1	3	3	5
$\rho^{\mathbb{H}}(\mathbb{H}^n)$	0	0	0	0	0	1	2

constructions using combinatorial methods see the recent work of Ognikyan [7]. The situation is completely different with complex and quaternionic vector fields; there is no explicit constructions that gives two or more linearly independent complex vector fields on $S(\mathbb{C}^n)$ and there is no known construction that gives even a single quaternionic vector field on $S(\mathbb{H}^n)$. In fact, the only known complex vector field on $S(\mathbb{C}^n)$ is the one given in Example 2.4.

From Theorem 2.5, we know that one can use m linearly independent complex (respectively, quaternionic) vector fields to obtain $2m$ linearly independent real (respectively, complex) vector fields. So, it is natural to ask if, in some way, it is possible to use linearly independent real (respectively, complex) vector fields to build some linearly independent complex or quaternionic (respectively, quaternionic) vector fields.

In the following theorem, we give necessary and sufficient conditions on linearly independent real (respectively, complex) vector fields to be linearly independent complex or quaternionic (respectively, quaternionic) vector fields. For simplicity, let $r_{\mathbb{C},i} = r_{\mathbb{C}} \circ \alpha_i \circ r_{\mathbb{C}}^{-1}$, $c_{\mathbb{H},j} = r_{\mathbb{C}} \circ \alpha_j \circ r_{\mathbb{H}}^{-1}$, and for each $t \in \{i, j, k\}$, let $r_{\mathbb{H},t} = r_{\mathbb{H}} \circ \alpha_t \circ r_{\mathbb{H}}^{-1}$.

- Theorem 4.1** (i) Suppose u_1, \dots, u_m are linearly independent real vector fields on $S(\mathbb{R}^{2n})$. Then $r_{\mathbb{C}}^{-1} \circ u_1 \circ r_{\mathbb{C}}, \dots, r_{\mathbb{C}}^{-1} \circ u_m \circ r_{\mathbb{C}}$ are linearly independent complex vector field on $S(\mathbb{C}^n)$ if and only if $u_1, \dots, u_m, r_{\mathbb{C},i} \circ u_1, \dots, r_{\mathbb{C},i} \circ u_m$ are linearly independent real vector fields on $S(\mathbb{R}^{2n})$.
- (ii) Suppose v_1, \dots, v_m are linearly independent complex vector fields on $S(\mathbb{C}^{2n})$. Then $c_{\mathbb{H}}^{-1} \circ v_1 \circ c_{\mathbb{H}}, \dots, c_{\mathbb{H}}^{-1} \circ v_m \circ c_{\mathbb{H}}$ are linearly independent quaternionic vector field on $S(\mathbb{H}^n)$ if and only if $v_1, \dots, v_m, c_{\mathbb{H},j} \circ v_1, \dots, c_{\mathbb{H},j} \circ v_m$ are linearly independent complex vector fields on $S(\mathbb{C}^{2n})$.
- (iii) Suppose u_1, \dots, u_m are linearly independent real vector fields on $S(\mathbb{R}^{4n})$. Then $r_{\mathbb{H}}^{-1} \circ u_1 \circ r_{\mathbb{H}}, \dots, r_{\mathbb{H}}^{-1} \circ u_m \circ r_{\mathbb{H}}$ are linearly independent quaternionic vector field on $S(\mathbb{H}^n)$ if and only if $u_1, \dots, u_m, r_{\mathbb{H},t} \circ u_1, \dots, r_{\mathbb{H},t} \circ u_m$, where $t \in \{i, j, k\}$, are linearly independent real vector fields on $S(\mathbb{R}^{4n})$.

Proof. Follows directly from Theorem 2.5.

In [3], Becker solved several important cases of the equivariant real vector fields problem, both the maximal number and the construction, on spheres with free group action by using the known Clifford algebras constructions of the non-equivariant real vector fields on spheres. As noted in [6] and [8], both sides of the equivariant complex and quaternionic vector fields problem on spheres with free group action is still completely open. By using a method similar to that used by Becker, the construction of the non-equivariant complex and quaternionic vector fields on spheres might lead to the solution of the equivariant complex and quaternionic vector fields problem on spheres with free group action.

References

- [1] J. F. Adams, Vector fields on spheres, *Ann. of Math.* **75** (3) (1962) 603-632.
- [2] J. F. Adams, G. Walker, On complex Stiefel manifolds, *Proc. Cambridge Philos. Soc.* **61** (1965) 81-103.
- [3] J.C. Becker, The span of spherical forms, *Amer. J. Math.* **94** (1972) 991-1026.
- [4] I. James, Cross Sections of Stiefel Manifolds, *Proc. London Math. Soc.* **8** (3) (1958) 536-547.
- [5] N. Mahammad, R. Piccinini, U. Suter, Some applications of topological K-theory, *North-Holland Math. Studies*, vol. 45, North Holland, Amsterdam, 1980.
- [6] M. Obiedat, Real, complex and quaternionic equivariant vector fields on spheres, *Topology Appl.* **153** (2006) 2182-2189.
- [7] A. A. Ognikyan, Combinatorial Construction of Tangent Vector Fields on Spheres, *Mathematical Notes* **83** (3) 539-553 (2008).
- [8] T. Önder, Equivariant cross sections of complex Stiefel manifolds, *Topology Appl.* **109** (2001) 107-125.

- [9] F. Sigrist, U. Suter, Cross sections of symplectic Stiefel manifolds, Trans. Amer. Math. Soc. **184** (1973) 247-259.
- [10] E. Thomas, Vector fields on manifolds, Bull. Amer. Math. Soc. 75 (1969) 643-683.
- [11] P. Zvengrowski, Canonical vector fields on spheres, Comment. Math. Helv. 43 (1968) 341-347.